

Geometric Phases and Mielnik's Evolution Loops¹

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Abstract.

The cyclic evolutions and associated geometric phases induced by time-independent Hamiltonians are studied for the case when the evolution operator becomes the identity (those processes are called *evolution loops*). We make a detailed treatment of systems having equally-spaced energy levels. Special emphasis is made on the potentials which have the same spectrum as the harmonic oscillator potential (the generalized oscillator potentials) and on their recently found coherent states.

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Since the appearance of Berry's work [1], much effort has been spent in studying geometric aspects of nonrelativistic quantum mechanics [2–8]. In particular, to *any* cyclic evolution of the vector state, $|\psi(\tau)\rangle = e^{i\phi}|\psi(0)\rangle$, there has been associated a *geometric phase*

$$\beta = \phi + \hbar^{-1} \int_0^\tau \langle \psi(t) | H(t) | \psi(t) \rangle dt, \quad (1)$$

where τ is the period of $|\psi(t)\rangle \in \mathcal{H}$, $\langle \psi(t) | \psi(t) \rangle = 1$, $\phi \in \mathbf{R}$, \mathcal{H} is the Hilbert space of vector states of the system, and $H(t)$ is the Hamiltonian [3]. β describes global curvature effects arising on the space of physical states, which is the projective space \mathcal{P} formed by the rays or the density operators $|\psi\rangle\langle\psi|$ instead of \mathcal{H} . Due to this curvature, the horizontal lifting (parallel transport) of the closed trajectory $|\psi(t)\rangle\langle\psi(t)| \in \mathcal{P}$ leads to a trajectory $|\psi_H(t)\rangle$ which is, in general, open on \mathcal{H} . The holonomy of this lifting is the Aharonov-Anandan geometric phase factor $e^{i\beta}$.

The phase β , determined up to a multiple of 2π , generalizes the Berry phase, which originally was defined just for *adiabatic* cyclic evolutions [1]. Further generalizations of β have been designed and can be found in the literature [3, 5–6]. There is also a lot of work dealing with the calculation of the geometric phases when the Hamiltonian is time dependent (either explicitly or implicitly through certain sets of time-dependent parameters [9–17]). In this paper, we will address the study of the geometric phases when the Hamiltonian is time independent, i.e. $H(t) = H$. This choice is done because, it seems to us, there is a widespread belief that the geometric phases appear only when the Hamiltonian is time dependent, which is wrong. This is, perhaps, motivated by the historical development of the subject and the following reasoning: the eigenstates $|E_n\rangle$ of H evolve according to $|E_n(t)\rangle = e^{-iE_n t/\hbar}|E_n\rangle$, where the E_n are the energy eigenvalues and n denotes a set of discrete subscripts. These evolutions are cyclic with period (arbitrary) τ and $\phi = -E_n\tau/\hbar$. Therefore, from (1), $\beta = 0$ for these states, and as usually the only cyclic states at hand for these systems are the eigenstates of the Hamiltonian, one is led to the wrong conclusion stated above. However, Aharonov and Anandan found nonnull geometric phases for a spin 1/2 in a constant homogeneous magnetic field [3]. The same will be true for any other two-level system described by a time-independent Hamiltonian [11, 15]. For nonspin systems it is possible to prove the existence of nontrivial geometric phases for the harmonic oscillator [15, 18] and some other physically interesting models (such as the localized states of an electron on a crystal [15]).

For independent reasons, in order to be used as the starting point for the techniques of “controlling” and “manipulating” the quantum systems, the *evolution loops* (EL) were proposed (although without this name at that time) in 1977 by Mielnik [19] and further developed by him in 1986 [20]. Those loops are specific dynamical processes induced either by time-dependent [19–21] or time-independent [22] Hamiltonians, for which the evolution operator $U(t)$ becomes the identity $\mathbf{1}$ (modulo a phase) for a certain time $\tau > 0$, i.e.:

$$U(\tau) = e^{i\phi} \mathbf{1}, \quad (2)$$

where $U(0) = \mathbf{1}$ (see also [23]). The EL are useful because when perturbed by some additional external fields, the system can be driven to attain any desired unitary operation on \mathcal{H} due to the accumulation of the small precessions of the distorted loop [19–21]. In the context of geometric phases a system performing an evolution loop is very interesting because *any* state becomes cyclic at $t = \tau$:

$$|\psi(\tau)\rangle = e^{i\phi} |\psi(0)\rangle. \quad (3)$$

Therefore, it could (we will show) have an associated nonnull geometric phase. We will restrict ourselves in this paper to the evaluation of the geometric phases associated to an evolution loop when the Hamiltonian is time independent.

Suppose one has a system with a time-independent Hamiltonian H whose evolution operator performs an evolution loop. Hence, any vector state $|\psi(t)\rangle$ comes back to itself at $t = \tau$ (see equations (2-3)), and its geometric phase can be easily evaluated because the evolution operator $U(t) = e^{-iHt/\hbar}$ commutes with H :

$$\beta = \phi + \hbar^{-1} \int_0^\tau \langle \psi(0) | U^\dagger(t) H U(t) | \psi(0) \rangle dt = \phi + \hbar^{-1} \tau \langle H \rangle, \quad (4)$$

where $\langle H \rangle = \langle \psi(0) | H | \psi(0) \rangle$. Expressing $|\psi(0)\rangle$ in terms of the basis $\{|E_m\rangle\}$, $|\psi(0)\rangle = \sum_m c_m |E_m\rangle$, with $c_m = \langle E_m | \psi(0) \rangle$, we find that (4) becomes:

$$\beta = \phi + \hbar^{-1} \tau \sum_m |c_m|^2 E_m. \quad (5)$$

Note that formulas (4-5) are applicable to the cyclic evolution of a vector state induced by any *time-independent* Hamiltonian regardless of whether or not the system performs an evolution loop. However, if the system has an evolution loop, then (4-5) will be valid for any initial condition. In particular, for $|\psi(0)\rangle = |E_n\rangle$, i.e., $c_m = \delta_{nm}$, it turns out that

$\phi = -E_n\tau/\hbar$, and hence $\beta = 0$. If at least two c_m 's are distinct from zero, however, the β associated to the corresponding cyclic state will be, in general, nontrivial (see [11], section 3.1).

There are in the literature some interesting systems whose time-independent Hamiltonian induces evolution loops [15, 18, 22 – 23]. Here, we will show the existence of an evolution loop for Hamiltonians whose spectrum consists of equally spaced energy levels of the form:

$$E_n = E_0 + n\Delta E, \quad (6)$$

where $\Delta E > 0$ is the constant spacing between the levels and E_0 is the ground state energy. The subscript $n \in \mathbf{Z}^+$ takes values in $[0, N]$, where N is finite if \mathcal{H} is finite-dimensional or infinite if \mathcal{H} is infinite-dimensional. The evolution operator of this system reads:

$$U(t) = \sum_{n=0}^N e^{-iE_n t/\hbar} |E_n\rangle\langle E_n|. \quad (7)$$

It is easy to see that the evolution loop is present at $\tau = 2\pi\hbar/\Delta E$:

$$U(\tau) = \sum_{n=0}^N e^{-i2\pi(E_0+n\Delta E)/\Delta E} |E_n\rangle\langle E_n| = e^{-i2\pi E_0/\Delta E} \mathbf{1}. \quad (8)$$

By comparing with (2), we obtain $\phi = -2\pi E_0/\Delta E$. Moreover, according to (4-5), the geometric phase for the cyclic state $|\psi(t)\rangle$ is:

$$\beta = 2\pi \frac{(\langle H \rangle - E_0)}{\Delta E} = 2\pi \sum_{n=1}^N n |c_n|^2 \geq 0. \quad (9)$$

Notice that the component c_0 of $|\psi(0)\rangle$ along the ground state $|E_0\rangle$ is not explicitly present in (9). If β is restricted (modulo 2π) to the interval $[0, 2\pi)$, then equation (9) admits the following interpretation: β measures the “energy excess” (in dimensionless units) of $\langle H \rangle$ above the nearest lower energy level E_k (see Figure 1). If E_k is given and $|\psi(0)\rangle$ is changed so that $E_k \leq \langle H \rangle < E_{k+1}$, then to the end $\beta = 0$ corresponds cyclic states with $\langle H \rangle = E_k$ (this includes in particular $|\psi(0)\rangle = |E_k\rangle$). To any other $\beta \in (0, 2\pi)$ corresponds cyclic states with $\langle H \rangle \neq E_k$ and vice versa (here necessarily $|\psi(0)\rangle \neq |E_k\rangle$).

One of the interesting systems with equally spaced energy eigenvalues for which our treatment can be applied is a spin j interacting with a constant homogeneous magnetic field \mathbf{B} , where $j > 0$ can be either integer or half-integer. Suppose, for simplicity, that

the magnetic field points in the z direction, $\mathbf{B} = B\mathbf{k}$. The spin Hamiltonian can be expressed as $H = -\mu\mathbf{J} \cdot \mathbf{B} = -\omega_c J_3$, where μ is the spin magnetic moment, \mathbf{J} is the spin operator whose components satisfy $[J_k, J_l] = i\hbar\epsilon_{klm}J_m$, with $k, l, m = 1, 2, 3$, and $\mu B = \omega_c > 0$ is the precession frequency of the spin around \mathbf{k} . We work in the basis $\{|j, m\rangle : J_3|j, m\rangle = m\hbar|j, m\rangle, -j \leq m \leq j\}$. Hence, $\dim(\mathcal{H}) = 2j + 1 = N + 1$. Due to the minus sign in the Hamiltonian, the identifications $\Delta E = \hbar\omega_c$, $E_0 = -j\hbar\omega_c$, and $|E_n\rangle = |j, j - n\rangle$ with $0 \leq n \leq 2j = N$ are consistent with equations (6-9). Therefore, the system performs an evolution loop and so any spin state evolves in a cyclic way, with an associated geometric phase given by equation (9). This is true, in particular, for the spin $j = 1/2$. In this case, it has become a convention to express the generic initial state in terms of the spherical angles θ, φ :

$$|\psi(0)\rangle = e^{-i\varphi/2} \cos(\theta/2)|1/2, 1/2\rangle + e^{i\varphi/2} \sin(\theta/2)|1/2, -1/2\rangle.$$

As the ground state in our notation is $|E_0\rangle = |1/2, 1/2\rangle$, the only coefficient contributing to the geometric phase is $c_1 = e^{i\varphi/2} \sin(\theta/2)$. Therefore, by applying (9), the geometric phase becomes the usual one [3]:

$$\beta = 2\pi|c_1|^2 = 2\pi \sin^2(\theta/2) = \pi(1 - \cos \theta).$$

Moreover, as is well known, a problem involving just two energy levels can be treated as a fictitious spin $1/2$ interacting with a homogeneous magnetic field [11, 15]; hence, taking care in making a judicious identification of the parameters, the same formulas to evaluate its geometric phases may be applied.

At this point, it is worth discussing a geometric interpretation applicable to systems with energy levels given by (6). It is easily understood for the spin-1/2 system of the previous example, for which the space of physical states (the projective space) coincides with the unit sphere S^2 on \mathbf{R}^3 . Any spin-1/2 state is precessing around the z axis, performing cyclic evolutions with a geometric phase which is, in general, distinct from zero. There are two states, however, for which the evolution is trivial: during the course of time they remain static at the north and south poles on S^2 , and correspond to the spin aligned along and in the opposite direction of the magnetic field. The geometric phase for both of them is zero. For a system with $N + 1$ equally-spaced energy levels, however, we have at hand a more interesting (and more complicated) situation: now, instead of having two static points on S^2 there are $N + 1$ points remaining static under the evolution on \mathcal{P}

(those associated to the $|E_n\rangle$, with N either finite or infinite). For each one of them the geometric phase is zero. Any other state will move across those points performing a more complicated (cyclic) evolution on \mathcal{P} with a nonnull geometric phase (in general) which can be easily evaluated using equation (9).

We proceed now to the analysis of another system having equally-spaced energy spectrum. It can be called the *generalized oscillator* (GO) because its energy levels are exactly the same as the ones of the harmonic oscillator Hamiltonian. The GO potentials were discovered by Abraham and Moses [24] using the Gelfand-Levitan formalism [25] and were generated by Mielnik [26] using a generalization of the well-known factorization method [27] (see also [28]). Because of its didactic value, we will point out some steps used in the generalized factorization to generate the GO potentials. We will work from now on in the coordinate representation with dimensionless units $\hbar = m = \omega = 1$.

The *classical* factorization method applied to the oscillator consists in expressing the Hamiltonian

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right) \quad (10)$$

as the two products

$$aa^\dagger = H + \frac{1}{2}, \quad a^\dagger a = H - \frac{1}{2}, \quad (11)$$

where a and a^\dagger are the ordinary ladder operators $a = (1/\sqrt{2})(d/dx + x)$, $a^\dagger = (1/\sqrt{2})(-d/dx + x)$ with $[a, a^\dagger] = \mathbf{1}$. The eigenfunctions and eigenvalues of the harmonic oscillator can be constructed using the relations

$$Ha^\dagger = a^\dagger(H + 1), \quad Ha = a(H - 1). \quad (12)$$

There is a normalized ground state $\psi_0(x)$ with eigenvalue $E_0 = 1/2$ which satisfies $a\psi_0(x) = 0 \Rightarrow \psi_0(x) \propto e^{-x^2/2}$, while the normalized eigenfunction $\psi_n(x)$ associated to the eigenvalue $E_n = n + 1/2$ is related to the ground state through:

$$\psi_n(x) = \frac{(a^\dagger)^n}{\sqrt{n!}} \psi_0(x). \quad (13)$$

The *generalized* factorization method [26] consists in looking for more general operators

$$b = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \beta(x) \right), \quad b^\dagger = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \beta(x) \right), \quad (14)$$

satisfying just the first one of relations (11):

$$bb^\dagger = H + \frac{1}{2}. \quad (15)$$

Hence, the unknown function $\beta(x)$ obeys the Riccati equation

$$\beta' + \beta^2 = 1 + x^2, \quad (16)$$

whose general solution is

$$\beta(x) = x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2} dy}, \quad \lambda \in \mathbf{R}. \quad (17)$$

Now, the point is that the product $b^\dagger b$ is no longer related to the harmonic oscillator Hamiltonian, but it leads to a new operator H_λ :

$$b^\dagger b = H_\lambda - \frac{1}{2}, \quad (18)$$

where

$$H_\lambda = -\frac{1}{2} \frac{d^2}{dx^2} + V_\lambda(x), \quad (19)$$

with

$$V_\lambda(x) = \frac{x^2}{2} - \frac{d}{dx} \left(\frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2} dy} \right) = \left(x + \frac{e^{-x^2}}{\lambda + \int_0^x e^{-y^2} dy} \right)^2 - \frac{x^2}{2}. \quad (20)$$

The requirement $|\lambda| > \sqrt{\pi}/2$ assures that $V_\lambda(x)$ has no singularities. The relationships analogous to (12) provide the way to obtain the eigenfunctions and eigenvalues of H_λ :

$$H_\lambda b^\dagger = b^\dagger (H + 1), \quad H b = b (H_\lambda - 1). \quad (21)$$

Hence, the states $\theta_n(x) = b^\dagger \psi_{n-1}(x)/\sqrt{n}$, $n = 1, 2, \dots$, are orthonormalized eigenfunctions of H_λ with eigenvalues $E_n = n + 1/2$. However, the set $\{\theta_n(x), n = 1, 2, \dots\}$ is not yet a basis of $L^2(\mathbf{R})$. There is a missing unit vector $\theta_0(x)$ which is orthogonal to all the vectors $\theta_n(x), n = 1, 2, \dots$. It turns out to be an eigenfunction of H_λ with eigenvalue $E_0 = 1/2$ satisfying $b\theta_0(x) = 0$, and taking the form:

$$\theta_0(x) \propto \exp \left(- \int_0^x \beta(y) dy \right). \quad (22)$$

As the set $\{\theta_n(x), n = 0, 1, 2, \dots\}$ forms a basis in $L^2(\mathbf{R})$, then $\{H_\lambda : |\lambda| > \sqrt{\pi}/2\}$ is a family of Hamiltonians distinct from the harmonic oscillator one but which has exactly

the same spectrum as the oscillator has. In the limit $|\lambda| \rightarrow \infty$, the harmonic oscillator potential is recovered, $V_\lambda(x) \rightarrow x^2/2$ when $|\lambda| \rightarrow \infty$.

All the relationships involving the evolution loops and the geometric phase (equations (6-9)) can be applied to the GO Hamiltonian (19-20) with $E_0 = 1/2$, $\Delta E = 1$, $\tau = 2\pi$, $\phi = -\pi$, and $N = \infty$. In particular, the geometric phase is $\beta = 2\pi(\langle H_\lambda \rangle - 1/2)$, and when applied to the wavefunctions of the basis $\{\theta_n(x), n = 0, 1, 2 \dots\}$ we recover again $\beta = 2n\pi$. Is there any other set of generic states of the GO potential for which we can evaluate explicitly the geometric phase? The answer turns out to be positive when considering the family of recently found coherent states for the GO Hamiltonian [29]. Here, we will present some details of its derivation (for work involving coherent states and their geometric phases see [9, 11, 15, 18, 30 – 32]).

In the construction of the coherent states of H_λ , denoted as $|z\rangle$ with $z \in \mathbf{C}$, we need to identify the “annihilation” and “creation” operators of the system. Because $b\theta_n(x) \propto \psi_{n-1}(x) \Rightarrow ab\theta_n(x) \propto \psi_{n-2}(x) \Rightarrow b^\dagger ab\theta_n(x) \propto \theta_{n-1}(x)$, and an obvious choice is:

$$A = b^\dagger ab, \quad A^\dagger = b^\dagger a^\dagger b. \quad (23)$$

The coherent states can be defined now as the eigenstates of the annihilation operator A with eigenvalues z , i.e. $A|z\rangle = z|z\rangle$. Expressing $|z\rangle$ in terms of the basis $\{|\theta_n\rangle, n = 0, 1, 2 \dots\}$, and substituting explicitly that expression in the previous one, we find the following family of coherent states (after normalization):

$$|z\rangle = \frac{1}{\sqrt{{}_0F_2(1, 2; |z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{n! \sqrt{(n+1)!}} |\theta_{n+1}\rangle, \quad (24)$$

where $|\theta_n\rangle$ is the ket representing the eigenfunction $\theta_n(x)$ and ${}_0F_2(1, 2; y)$ is a generalized hypergeometric function defined by:

$${}_0F_2(\alpha, \beta; y) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+n)\Gamma(\beta+n)} \frac{y^n}{n!}, \quad (25)$$

with $\Gamma(\cdot)$ the Gamma function. To each value $z \neq 0$ corresponds one and only one coherent state. However, $z = 0$ is a doubly degenerate eigenvalue of A with two orthogonal eigenvectors which will be denoted $|\theta_0\rangle$ and $|z = 0\rangle = |\theta_1\rangle$. By choosing an appropriate measure in the complex plane, it can be shown that the set $\{|\theta_0\rangle, |z\rangle\}$ is complete in \mathcal{H} .

The relationships presented so far are sufficient for our purpose of evaluating the geometric phase. To this end, we need to find the expected value of H_λ in the state $|z\rangle$. A direct calculation leads to:

$$\langle H_\lambda \rangle = \langle z | H_\lambda | z \rangle = 1/2 + \frac{{}_0F_2(1, 1; |z|^2)}{{}_0F_2(1, 2; |z|^2)}. \quad (26)$$

Finally, substituting (26) in the equation for β , we obtain the following expression for the geometric phase β_{GCS} of the generalized coherent state:

$$\beta_{GCS} = 2\pi \frac{{}_0F_2(1, 1; |z|^2)}{{}_0F_2(1, 2; |z|^2)}. \quad (27)$$

To have an idea of the behaviour of β_{GCS} , we plot it versus $\text{Re}(z) \times \text{Im}(z)$ in Figure 2. As we can see, the geometric phase is independent of λ and depends on z in a quite different way compared with that of a standard coherent state (SCS) of the harmonic oscillator, for which $\beta_{SCS} = 2\pi|z|^2$ [15, 18] (see also Figure 2). This occurs because the generalized coherent states discussed in [29] do not tend to the standard ones when $\lambda \rightarrow \infty$ even though the generalized potential tends to the harmonic oscillator potential in this limit. A deeper analysis shows that the difference rests on the fact that in this limit the annihilation operator $A_\infty \equiv \lim_{\lambda \rightarrow \infty} A = a^\dagger a^2$ is distinct from the standard one a . The generalized coherent states, however, could be useful in future applications because the product of the uncertainty of the \hat{X} and \hat{P} operators for these states is almost minimum in this limit $1/2 \leq \lim_{|\lambda| \rightarrow \infty} \Delta \hat{X} \Delta \hat{P} \leq 3/2$. The question of whether or not there is a family of coherent states of H_λ tending to the standard ones when $\lambda \rightarrow \infty$, the geometric phases included, is open.

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Fig.1 Schematic representation of the $N + 1$ energy levels for a system with equally-spaced spectrum. If the geometric phase β is restricted to the interval $[0, 2\pi)$, then it can be interpreted as an energy excess of the system with respect to its nearest energy level E_k (the nearest below) in dimensionless units.

Fig.2 The geometric phases associated to the standard coherent states of the harmonic oscillator (β_{SCS}) and the coherent states of the generalized oscillator (β_{GCS}) as functions of the complex variable z . The minimum values of β_{SCS} and β_{GCS} are 0 and 2π respectively, both at $z = 0$. The missing sections in both surfaces were removed to show the behaviour close to the minimum.

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